Non-Commutative Harmonic and Subharmonic Polynomials

J. William Helton, Daniel P. McAllaster and Joshua A. Hernandez

UCSD Department of Mathematics
La Jolla, CA. 92093

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Abstract

The paper introduces a notion of the Laplace operator of a polynomial p in noncommutative variables $x = (x_1, \ldots, x_g)$. The Laplacian $Lap[p, h]$ of p is a polynomial in x and in a noncommuting variable h. When all variables commute we have $Lap[p, h] = h^2 \Delta_x p$ where $\Delta_x p$ is the usual Laplacian. A symmetric polynomial in symmetric variables will be called *harmonic* if $Lap[p, h] = 0$ and *subharmonic* if the polynomial $q(x, h) := Lap[p, h]$ takes positive semidefinite matrix values whenever matrices $X_1, \ldots, X_g, H$ are substituted for the variables $x_1, \ldots, x_g, h$. In this paper we classify all homogeneous symmetric harmonic and subharmonic polynomials in two symmetric variables. We find there are not many of them: for example, the span of all such subharmonics of any degree higher than 4 has dimension 2 (if odd degree) and 3 (if even degree). Hopefully, the approach here will suggest ways of defining and analyzing other partial differential equations and inequalities.

1 Introduction

In the introduction we shall make essential definitions, then state our main results. The rest of the paper proves them.

1.1 Definitions

1.1.1 Non-Commutative Polynomials

A non-commutative monomial m of degree d on the free variables $(x_1, \ldots, x_g)$ is a product $x_{a_1} x_{a_2} \cdots x_{a_d}$ of these variables, corresponding to a unique sequence of a_i of nonnegative integers, $1 \le a_i \le g$. We abbreviate this $m = x^w$, where w is the d-tuple $(a_1, \ldots, a_d)$. The set of all monomials in $(x_1, \ldots, x_g)$ is denoted as $\mathcal{M}$ and the set of indexes w is denoted $\mathcal{W}$. Some notation is:

$\lvert w \rvert = d$	the length of w
$(w)_i = a_i$	the i^{th} entry of w
$w^T = (a_d, \ldots, a_1)$	the transpose of w
$\phi = ()$	the empty word (word of length zero)

The space of non-commutative polynomials $p(x) = p(x_1, \ldots, x_g)$ with real coefficients is denoted $\mathbb{R}\langle x \rangle$ and we express p as

$$p(x) = \sum_{m \in \mathcal{M}} A_m \, m.$$

An example of a non-commutative polynomial is

$$p(x) = p(x_1, x_2) = x_1^2 \, x_2 \, x_1 + x_1 \, x_2 \, x_1^2 + x_1 \, x_2 - x_2 \, x_1 + 7$$

(in commutative variables, this would be equivalent to $2x_1^3 x_2 + 7$).

The transpose of a monomial $m = x^w$ is defined to be $m^T = x^{w^T}$. The transpose of a polynomial p, denoted p^T, is defined by $p(x) = \sum_{m \in \mathcal{M}} A_m \, m^T$ and has the following properties:

(1) $(p^T)^T = p$
(2) $(p_1 + p_2)^T = p_1^T + p_2^T$
(3) $(\alpha p)^T = \alpha p^T \qquad (\alpha \in \mathbb{R})$
(4) $(p_1 \, p_2)^T = p_2^T \, p_1^T$.

In this paper, we shall consider primarily polynomials in symmetric variables. That is, we consider variables x_i where $x_i^T = x_i$. Then monomials satisfy $(x_{a_1} \ldots x_{a_{d-1}})^T = x_{a_{d-1}} \ldots x_{a_1}$, which in other notation is $(x^w)^T = x^{w^T}$. Symmetric (or self-adjoint) polynomials are those that are equal to their transposes.

1.1.2 Evaluating Noncommutative Polynomials

Let $(\mathbb{R}^{n\times n}_{sym})^g$ denote the set of g-tuples $(X_1, \ldots, X_g)$ of real symmetric $n \times n$ matrices. We shall be interested in evaluating a polynomial $p(x) = p(x_1, \ldots, x_g)$ that belongs to $\mathbb{R}\langle x \rangle$ at a tuple $X = (X_1, \ldots, X_g) \in (\mathbb{R}^{n\times n}_{sym})^g$. In this case $p(X)$ is also an $n \times n$ matrix and the involution on $\mathbb{R}\langle x \rangle$ that was introduced earlier is compatible with matrix transposition, i.e.,

$$p^T(X) = p(X)^T,$$

where $p(X)^T$ denotes the transpose of the matrix $p(X)$. When $X \in (\mathbb{R}^{n\times n}_{sym})^g$ is substituted into p the constant term $p(0)$ of $p(x)$ becomes $p(0)I_n$. Thus, for example,

$$p(x) = 3 + x_1^2 + 5x_2^3 \quad \Longrightarrow \quad p(X) = 3I_n + X_1^2 + 5X_2^3.$$

A symmetric polynomial $p \in \mathbb{R}\langle x\rangle$ is **matrix positive** if $p(X)$ is a positive semidefinite matrix for each tuple $X = (X_1, \dots, X_g) \in (\mathbb{R}^{n\times n}_{sym})^g$. We emphasize that throughout this paper, unless otherwise noted, $x_1, x_2, \dots, x_n$ stand for variables and $X_1, X_2, \dots, X_n$ stand for matrices (usually symmetric).

1.1.3 Non-Commutative Differentiation

For our non-commutative purposes, we take directional derivatives in x_i with regard to an indeterminate direction parameter h.

$$D[p(x_1, \dots, x_g), x_i, h] := \frac{d}{dt}[p(x_1, \dots, (x_i + th), \dots, x_g)]_{|t=0}. \tag{1}$$

We say that this is the directional derivative of $p(x) = p(x_1, \dots, x_g)$ in x_i in the direction h. Note it is linear in h. For a detailed formal definition see [HMV06], for more examples see [CHSY03].

Example 1.1. The directional derivative

$$\begin{aligned} D[x_1^2\, x_2, x_1, h] &= \tfrac{d}{dt}[(x_1 + th)^2 x_2]_{|t=0} \\ &= \tfrac{d}{dt}[x_1^2\, x_2 + th\, x_1\, x_2 + tx_1\, h\, x_2 + t^2h^2\, x_2]_{|t=0} \\ &= [h\, x_1\, x_2 + x_1\, h\, x_2 + 2th^2\, x_2]_{|t=0} \\ &= h\, x_1\, x_2 + x_1\, h\, x_2. \end{aligned}$$

As this example shows, the directional derivative of p on x_i in the direction h is the sum of the terms produced by replacing one instance of x_i with h.

Lemma 1.1. *The directional derivative of NC polynomials is linear,*

$$D[a\, p(x) + b\, q(x), x_i, h] = a\, D[p(x), x_i, h] + b\, D[q(x), x_i, h]$$

and respects transposes

$$D[p(x)^T, x_i, h] = D[p(x), x_i, h]^T.$$

Proof. Straighforward. ☐

1.1.4 Non-Commutative Laplacian and Subharmonicity

The Laplacian of a NC polynomial $p(x)$ is defined as:

$$Lap[p, h] := \sum_{i=1}^{g} D[D[p(x), x_i, h],\ x_i, h] \tag{2}$$

$$= \sum_{i=1}^{g} \frac{d^2}{dt^2}[p(x_1, \dots, (x_i + th), \dots, x_g)]_{|t=0}. \tag{3}$$

Our notation is slightly inconsistent (but has advantages) in that the single letter x stands for g variables $x_1, \dots, x_g$ while h is a single variable. Note that Lap is linear in h. An NC polynomial is called **harmonic** if its Laplacian is zero, and **subharmonic** if its Laplacian is matrix-positive and **purely subharmonic** is used to describe a polynomial which is subharmonic but not harmonic - that is, having a nonzero, matrix-positive Laplacian.

Specialization of $Lap[p, h]$, to commutative variables, is $h^2\Delta(p)$ where $\Delta(p)$ is the standard Laplacian, namely, $\Delta(p) := \sum_{i=1}^{g} \partial_{x_i x_i} p(x)$. Here $p : \mathbb{R}^n \to \mathbb{R}$.

1.2 Classification of Harmonics and Subharmonics in Two Variables

For our special homogeneous polynomials on two variables, define

$$\gamma := x_1 + i\, x_2 \tag{4}$$

where i is the imaginary number.

Theorem 1. *The homogeneous noncommutative polynomials in two symmetric variables which are*

(1a.) *harmonic of degree $d > 2$ are exactly the linear combinations of*

$$\mathrm{Re}(\gamma^d) \quad \textit{and} \quad \mathrm{Im}(\gamma^d),$$

(1b.) *harmonic of degree $d = 2$ are exactly the linear combinations of*

$$\mathrm{Re}(\gamma^d) \quad \textit{and} \quad \mathrm{Im}(\gamma^d) \quad \textit{and} \quad x_1 x_2,$$

(note this includes $x_2 x_1$),

(2a) *subharmonic of degree $2d$ with $d > 2$, are exactly the linear combinations:*

$$\begin{aligned} &c_0[\mathrm{Re}(\gamma^d)]^2 \;+\; c_1\,\mathrm{Re}(\gamma^{2d}) \;+\; c_2\,\mathrm{Im}(\gamma^{2d}) \\ &\quad = c_0[\mathrm{Im}(\gamma^d)]^2 \;+\; (c_0 + c_1)\;\mathrm{Re}(\gamma^{2d}) \;+\; c_2\,\mathrm{Im}(\gamma^{2d}) \end{aligned} \tag{5}$$

where $c_0 \geq 0$,

(2b) *symmetric subharmonics of degree 4, are exactly the linear combinations:*

$$\begin{aligned} f \;=\; & B_1(x_1^4 - x_1^2\,x_2^2 - x_2^2\,x_1^2 + x_2^4) + B_2(x_1^3\,x_2 + x_2\,x_1^3 - x_2\,x_1\,x_2^2 - x_2^2\,x_1\,x_2) \\ & + B_3(x_1^2\,x_2\,x_1 + x_1\,x_2\,x_1^2 - x_1\,x_2^3 - x_2^3\,x_1) + B_4(x_1\,x_2\,x_1\,x_2 + x_2\,x_1\,x_2\,x_1) \\ & + B_5\,x_1\,x_2^2\,x_1 + B_6 x_2\,x_1^2\,x_2 \end{aligned} \tag{6}$$

with coefficients satisfying the inequalities:

$$(III) \implies (B_1 + B_6)(B_1 + B_5) > (B_3 - B_2)^2 + (B_1 + B_4)^2 \;\; \textit{and} \tag{7}$$

$$(I) \implies \quad B_1 + B_6 > 0 \quad (\textit{or, equivalently} \quad B_1 + B_5 > 0). \tag{8}$$

(2c) *All subharmonics of degree 2 are,*

$$A_1\,x_1^2 + A_2\,x_2^2 + A_3\,x_1 x_2 + A_4 x_2 x_1$$

with $A_1 + A_2 \geq 0$.

(3) *Pure subharmonics of odd degree do not exist.*

Note: all of these functions except for $x_1 x_2$ and $x_2 x_1$ in (1b) and in (2c) are symmetric.

Proof. Most of the remainder of this paper is focused on proving this theorem. The proofs for the parts of the theorem are as follows:

Part of theorem	Section of the proof
(1a.)	4.2.1
(1b.)	4.2.2
(2a.)	2.4
(2b.)	3.3 also Remark 2
(2c.)	4.2.2
(3.)	2.4

Remark 1. The following degree 3 polynomial p is unusual in that there is a region of X_1, X_2 where $Lap(p)$ is positive, but $Lap(p)$ is not positive everywhere:

$$A_1\,(x_1^3 - x_1\,x_2^2 - x_2\,x_1^2) + A_2\,x_2\,x_1\,x_2 + A_3\;x_1\,x_2\,x_1 + A_4\,(x_2^3 - x_1^2\,x_2 - x_2\,x_1^2)$$

For this the *region of subharmonicity* is $(A_1 + A_2)x_1 + (A_3 + A_4)x_2 > 0$ and the *region of harmonicity* is $A_1 + A_2 = A_3 + A_4 = 0$. Of course, there is no homogeneous polynomial of degree three which is subharmonic over all values of x_1 and x_2.

1.3 Subharmonics are All Built from Harmonics

Our second main result is a general fact which holds in any number of variables:

Theorem 2. *Assume the harmonic polynomials homogeneous of degree $\frac{d}{2}$ have a basis $\gamma_1, \cdots, \gamma_k$ with the independence property: there is a monomial w_j in γ_j which does not occur in the other $\gamma_1, \cdots, \gamma_k$. If p is a homogeneous symmetric subharmonic polynomial of degree even d, then p has the form*

$$p = \sum_{i}^{finite} c_i R_i^T R_i$$

for some homogeneous harmonic functions R_j of degree $\frac{d}{2}$ and real numbers c_j.

Because of this, knowing all homogeneous subharmonics will likely occur once the harmonics are classified.

Proof. The proof is found in §4.1.1.

1.4 Comparison with Commutative Subharmonic Polynomials

The study of harmonic and subharmonic polynomials in commuting variables is classical. Harmonic commuting polynomials are classified in any number of variables and the have a close correspondence to spherical harmonics. A good reference on this is [HT92] §2.4, pp. 110-113.

For two commuting variables, the homogeneous harmonic polynomials are those of the form,

$$Re(x_1 + Ix_2)^n \quad and \quad Im(x_1 + Ix_2)^n,$$

so the commuting and noncommuting case are exactly parallel.

1.5 Related Topics and Motivation

Non-Commutative Convexity The non-commutative Hessian is defined as:

$$\mathrm{NCHes}[p(x_1,\dots,x_g),\{x_1,\eta_1\},\dots,\{x_g,\eta_g\}] := \frac{d^2}{dt^2}[p(x_1+t\eta_1,\dots,x_g+t_g\eta_g)]_{|t=0}.$$

Note that this is composed of several independent direction parameters, η_i and that if p is a polynomial, then its Hessian is a polynomial in x and η which is homogeneous of degree 2 in η.

A non-commutative polynomial is considered **convex** wherever its Hessian is matrix-positive.

A polynomial $p(x) = p(x_1,\dots,x_d)$ is **geometrically convex** if and only if, for every $X, Y \in (\mathbb{R}^{n\times n}_{sym})^g$,

$$\frac{1}{2}\big(p(X)+p(Y)\big) - p\left(\frac{X+Y}{2}\right)$$

is positive-semidefinite. It is proved in [HM98] that convexity is equivalent to geometric convexity. A crucial fact regarding these polynomials (see [HM04]) is that they are all of degree two or less. Some excellent papers on noncommutative convexity are [HT06] [Han97].

The commutative analog of this "directional" Hessian is the quadratic function

$$H(p)\begin{pmatrix}\eta_1\\ \vdots\\ \eta_g\end{pmatrix}\cdot\begin{pmatrix}\eta_1\\ \vdots\\ \eta_g\end{pmatrix} \tag{9}$$

where $H(p)$ is the Hessian matrix:

$$\begin{pmatrix}\partial_{x_1x_1}p(x) & \cdots & \partial_{x_1x_g}p(x)\\ \vdots & \ddots & \vdots\\ \partial_{x_gx_1}p(x) & \cdots & \partial_{x_gx_g}p(x)\end{pmatrix}. \tag{10}$$

If this Hessian is positive semidefinite at all $(x_1,\dots,x_g)$, then f is said to be convex.

Non-Commutative Algebra in Engineering Inequalities, involving polynomials in matrices and their inverses, and associated optimization problems have become very important in engineering. When such polynomials are matrix convex, local minima are global. This is extremely important in applications. Also, interior point numerical methods apply well to these. In the last few years, the approaches that have been proposed in the field of optimization and control theory based on linear matrix inequalities and semidefinite programming have become very important and promising, since the same framework can be used for a large set of problems. Matrix inequalities provide a nice setup for many engineering and related problems, and if they are convex the optimization problem is well behaved and interior point methods provide efficient algorithms which are effective on moderate sized problems. Unfortunately, the class of matrix convex noncommutative polynomials is very small; as already mentioned they are all of degree two or less [HM04].

Our original interest in subharmonic polynomials was to analyze conditions similar to convexity, though not as restrictive, in the hopes of finding much broader classes which still had nice properties. What we found (as reported here) was that subharmonic polynomials are (in two) variables a highly restricted class.

Noncommutative Analysis This article would come under the general heading of "free analysis", since the setting is a noncommutative algebra whose generators are "free" of relations. This is a

burdgeoning area, of which free probability is currently the largest component. The interested reader is referred to the web site [SV06] of American Institute of Mathematics, in particular it gives the findings of the AIM workshop in 2006 on free analysis.

A fairly expository article describing noncommutative convexity, noncommutative semialgebraic geometry and relations to engineering is [HP06].

2 Existence Proofs

Now we set about to prove Theorem 1. In this section, we show that the polynomials claimed to be harmonic and subharmonic are indeed. In section 4, we show that these are the only posibilities.

2.1 Product Rules for Derivatives

To begin with, we will build up facts about derivatives.

2.1.1 Product Rule for First Derivatives

Lemma 2.1. *The product rule for the directional derivative of NC polynomials is*

$$D[p_1\, p_2, x_i, h] = D[p_1, x_i, h]\, p_2 \;+\; p_1\, D[p_2, x_i, h].$$

Proof. The directional derivative $D[m, x_i, h]$ of a product $m = m_1 m_2$ of non-commutative monomials m_1 and m_2 is the sum of terms produced by replacing one instance of x_i in m by h. This sum can be divided into two parts:

μ_1, the sum of terms whose h lie in the first $|m_1|$ letters, i.e. $D[m_1, x_i, h]m_2$

μ_2, the sum of terms whose h lie in the last $|m_2|$ letters, i.e. $m_1 D[m_2, x_i, h]$.

Therefore

$$D[m_1 m_2, x_i, h] = D[m_1, x_i, h]m_2 + m_1 D[m_2, x_i, h].$$

We can extend this product rule to the product of any two non-commutative polynomials p_1 and p_2 as follows.

$$\begin{aligned}
D[p_1\, p_2, x_i, h] &= D\Big[\Big(\sum_{m_1 \in \mathcal{W}_{p_1}} A_{m_1} m_1\Big)\Big(\sum_{m_2 \in \mathcal{W}_{p_2}} A_{m_2} m_2\Big), x_i, h\Big] \\
&= \sum_{m_1 \in \mathcal{W}_{p_1}} \sum_{m_2 \in \mathcal{W}_{p_2}} A_{m_1} A_{m_2} D[m_1\, m_2, x_i, h] \\
&= \sum_{m_1 \in \mathcal{W}_{p_1}} \sum_{m_2 \in \mathcal{W}_{p_2}} A_{m_1} A_{m_2} D[m_1, x_i, h]\, m_2 + A_{m_1} A_{m_2}\, m_1\, D[m_2 x_i, h] \\
&= D\Big[\sum_{m_1 \in \mathcal{W}_{p_1}} A_{m_1} m_1, x_i, h\Big] \sum_{m_2 \in \mathcal{W}_{p_2}} A_{m_2} m_2 + \sum_{m_1 \in \mathcal{W}_{p_1}} A_{m_1} m_1\, D\Big[\sum_{m_2 \in \mathcal{W}_{p_2}} A_{m_2} m_2, x_i, h\Big] \\
&= D[p_1, x_i, h]\, p_2 + p_1\, D[p_2, x_i, h]. \qquad (11)
\end{aligned}$$

□

2.1.2 The Laplacian of a Product

We now prove Theorem 1 part (2b).

Lemma 2.2. *The product rule for the Laplacian of NC polynomials is*

$$Lap[p_1\, p_2, h] = Lap[p_1, h]\, p_2 + p_1\, Lap[p_2, h] + 2\sum_{i=1}^{g}\big(D[p_1, x_i, h]\, D[p_2, x_i, h]\big).$$

As a consequence if p is harmonic, then

$$Lap[p^T\, p, h] = 2\sum_{i=1}^{g}\big(D[p, x_i, h]^T\, D[p, x_i, h]\big).$$

Proof.

$$\begin{aligned}
Lap[p_1\, p_2, h] &= \sum_{i=1}^{g} D[D[p_1\, p_2, x_i, h], x_i, h] \\
&= \sum_{i=1}^{g}\big(D[p_1\, D[p_2, x_i, h] + D[p_1, x_i, h]\, p_2,\ x_i, h]\big) \\
&= \sum_{i=1}^{g}\big(p_1\, D[D[p_2, x_i, h], x_i, h] + D[D[p_1, x_i, h], x_i, h]\, p_2 \\
&\qquad + 2D[p_1, x_i, h], D[p_2, x_i, h]\big) \\
&= Lap[p_1, h]\, p_2 + p_1\, Lap[p_2, h] + 2\sum_{i=1}^{g}\big(D[p_1, x_i, h]\, D[p_2, x_i, h]\big).
\end{aligned}$$

□

2.2 Formulas Involving γ^d and its Derivatives

Recall from (4) that $\gamma := x_1 + ix_2$.

Note that $\gamma = \gamma^T$ and therefore that $\gamma^d = (\gamma^d)^T$. So

$$\Big(\mathrm{Re}(\gamma^d)\Big)^T = \mathrm{Re}((\gamma^d)^T) = \mathrm{Re}(\gamma^d) \text{ and } \Big(\mathrm{Im}(\gamma^d)\Big)^T = \mathrm{Im}((\gamma^d)^T) = \mathrm{Im}(\gamma^d).$$

This proves the (last) assertion in Theorem 1 that all but a few subharmonics on our list are symmetric.

Lemma 2.3. *The derivatives of of γ^d exhibit the following symmetries.*

$$D[\mathrm{Re}(\gamma^d), x_1, h] = D[\mathrm{Im}(\gamma^d), x_2, h]$$

and

$$D[\mathrm{Re}(\gamma^d), x_2, h] = -D[\mathrm{Im}(\gamma^d), x_1, h].$$

Proof. The proof proceeds by induction. To begin, it is easily seen that

$$\begin{aligned}
D[\mathrm{Re}(\gamma), x_1, h] &= D[\mathrm{Im}(\gamma), x_2, h] = h \\
D[\mathrm{Im}(\gamma), x_1, h] &= -D[\mathrm{Re}(\gamma), x_2, h] = 0.
\end{aligned}$$

Assume that

$$\begin{aligned} D[\mathrm{Re}(\gamma^{d-1}), x_1, h] &= D[\mathrm{Im}(\gamma^{d-1}), x_2, h] \\ D[\mathrm{Im}(\gamma^{d-1}), x_1, 1] &= -D[\mathrm{Re}(\gamma^{d-1}), x_2, h]. \end{aligned}$$

Then

$$\begin{aligned} &D[\,\mathrm{Re}(\gamma^{d}), x_1, h] = D[x_1\mathrm{Re}(\gamma^{d-1}) - x_2\mathrm{Im}(\gamma^{d-1}), x_1, h] \\ &\quad = x_1\, D[\mathrm{Re}(\gamma^{d-1}), x_1, h] + h\,\mathrm{Re}(\gamma^{d-1}) - x_2\, D[\mathrm{Im}(\gamma^{d-1}), x_1, h] \end{aligned}$$

$$\begin{aligned} &D[\mathrm{Im}(\gamma^{d}), x_2, h] = D[x_1\mathrm{Im}(\gamma^{d-1}) + x_2\mathrm{Re}(\gamma^{d-1}), x_2, h] \\ &\quad = x_1\, D[\mathrm{Im}(\gamma^{d-1}), x_2, h] + x_2\, D[\mathrm{Re}(\gamma^{d-1}), x_2, h] + h\,\mathrm{Re}(\gamma^{d-1}), \end{aligned}$$

so

$$D[\mathrm{Re}(\gamma^{d}), x_1, h] = D[\mathrm{Im}(\gamma^{d}), x_2, h]$$

which satisfies the first half of our inductive hypothesis. For the next half compute

$$\begin{aligned} &D[\,\mathrm{Re}(\gamma^{d}), x_2, h] = D[x_1\mathrm{Re}(\gamma^{d-1}) - x_2\mathrm{Im}(\gamma^{d-1}), x_2, h] \\ &\quad = x_1\, D[\mathrm{Re}(\gamma^{d-1}), x_2, h] - x_2\, D[\mathrm{Im}(\gamma^{d-1}), x_2, h] - h\,\mathrm{Im}(\gamma^{d-1}) \end{aligned}$$

$$\begin{aligned} &D[\mathrm{Im}(\gamma^{d}), x_1, h] = D[x_1\mathrm{Im}(\gamma^{d-1}) + x_2\mathrm{Re}(\gamma^{d-1}), x_1, h] \\ &\quad = x_1\, D[\mathrm{Im}(\gamma^{d-1}), x_1, h] + h\,\mathrm{Im}(\gamma^{d-1}) + x_2\, D[\mathrm{Re}(\gamma^{d-1}), x_2, h], \end{aligned}$$

so

$$D[\mathrm{Re}(\gamma^{d}), x_2, h] = -D[\mathrm{Im}(\gamma^{d}), x_1, h].$$

□

2.3 Harmonics *degree* > 2: Proof of Theorem 1 part (1)

Our proof will proceed by induction. Since the Laplacian of words of length 1 is zero,

$$Lap[\mathrm{Re}(\gamma), h] = Lap[x_1, h] = 0 \quad \text{and} \quad Lap[\mathrm{Im}(\gamma), h] = Lap[x_2, h] = 0.$$

Now, assume that

$$Lap[\mathrm{Re}(\gamma^{d-1}), h] = Lap[\mathrm{Im}(\gamma^{d-1}), h] = 0. \tag{12}$$

Pushing ahead,

$$\mathrm{Re}(\gamma^{d}) = \mathrm{Re}((x_1 + ix_2)\,\gamma^{d-1}) = x_1\,\mathrm{Re}(\gamma^{d-1}) - x_2\,\mathrm{Im}(\gamma^{d-1}) \tag{13}$$

$$\mathrm{Im}(\gamma^{d}) = \mathrm{Im}((x_1 + ix_2)\,\gamma^{d-1}) = x_1\,\mathrm{Im}(\gamma^{d-1}) + x_2\,\mathrm{Re}(\gamma^{d-1}). \tag{14}$$

Applying our product rule to (13):

$$\begin{aligned}
Lap[\mathrm{Re}(\gamma^d), h] &= Lap[x_1 \,\mathrm{Re}(\gamma^{d-1}), h] - Lap[x_2 \,\mathrm{Im}(\gamma^{d-1}), h] \\
&= x_1 \, Lap[\mathrm{Re}(\gamma^{d-1}), h] + Lap[x_1, h] \,\mathrm{Re}(\gamma^{d-1}) \\
&\quad + 2D[x_1, x_1, h] \, D[\mathrm{Re}(\gamma^{d-1}), x_1, h] \\
&\quad + 2D[x_1, x_2, h] \, D[\mathrm{Re}(\gamma^{d-1}), x_2, h] \\
&\quad - x_2 \, Lap[\mathrm{Im}(\gamma^{d-1}), h] - Lap[x_2, h] \,\mathrm{Im}(\gamma^{d-1}) \\
&\quad - 2D[x_2, x_1, h] \, D[\mathrm{Im}(\gamma^{d-1}), x_1, h] \\
&\quad - 2D[x_2, x_2, h] \, D[\mathrm{Im}(\gamma^{d-1}), x_2, h].
\end{aligned}$$

Use (12) and (12) to obtain that the $Lap[]$ terms are 0, and that "cross partials are 0" to get

$$Lap[\mathrm{Re}(\gamma^d), h] = h \, D[\mathrm{Re}(\gamma^{d-1}), x_1, h] - h \, D[\mathrm{Im}(\gamma^{d-1}), x_2, h].$$

By symmetry Lemma 2.3, this means

$$\mathrm{Re}(Lap[\gamma^d, h]) = Lap[\mathrm{Re}(\gamma^d), h] = 0.$$

By a similar argument, applying the product rule to (14),

$$\mathrm{Im}(Lap[\gamma^d, h]) = Lap[\mathrm{Im}(\gamma^d), h] = 0.$$

□

2.4 Subharmonics *degree* > 4: Proof of Theorem 1 (2a.)

The product rule for the Laplacian of harmonics in Lemma 2.2 says $Lap[\big(\mathrm{Re}(\gamma^d)\big)^2, h]$ is a sum of squares. Thus we have shown $\big(\mathrm{Re}(\gamma^d)\big)^2$ is subharmonic.

Now we prove the formula (5) relating subharmonics. We use

$$\begin{aligned}
\gamma^{2d} &= (\mathrm{Re}(\gamma^d) + i \,\mathrm{Im}(\gamma^d))^2 \\
&= (\mathrm{Re}(\gamma^d))^2 - (\mathrm{Im}(\gamma^d))^2 + i(\mathrm{Re}(\gamma^d)\mathrm{Im}(\gamma^d) + \mathrm{Im}(\gamma^d) \,\mathrm{Re}(\gamma^d)).
\end{aligned}$$

Therefore

$$\mathrm{Re}(\gamma^{2d}) = \Big(\mathrm{Re}(\gamma^d)\Big)^2 - \Big(\mathrm{Im}(\gamma^d)\Big)^2.$$

So

$$\begin{aligned}
&c_0 \Big[\mathrm{Re}(\gamma^d)\Big]^2 + c_1 \,\mathrm{Re}(\gamma^{2d}) \; + \; c_2 \,\mathrm{Im}(\gamma^{2d}) \\
&\qquad = c_0 \Big[\mathrm{Im}(\gamma^d)\Big]^2 \; + \; (c_0 + c_1) \,(\mathrm{Re}(\gamma^{2d})) + c_2 \,\mathrm{Im}(\gamma^{2d}),
\end{aligned}$$

which is (5). □

Up to this point we have handled subharmonics of even degree. To see that there are no pure subharmonics of odd degree, note that the Laplacian $L(x)$ of an odd degree polynomial is itself an odd degree polynomial which is matrix-positive. Consider $L(tx)$ as $t \in \mathbb{R}$ approaches $\pm\infty$. Since the highest order terms dominate, the signs of these limits are opposite. Thus the highest order terms are 0.

3 Classification when Degree is Four or Less

We handle now what appear to be special cases which are exceptions to the general degree > 4 theorem.

3.1 The Matrix Representation

Important in our proofs for polynomials of low degree is a representation of polynomials $q(x_1, \cdots, x_g)[h]$ which are homogeneous of degree 2 in h. Recall that often x stands for $(x_1, \cdots, x_g)$ and h is a single variable. In our notation $q(x)[h]$ we use [ ] to distinguish the variable which is of degree 2.

Any NC symmetric polynomial q in symmetric variables quadratic in h can be written

$$q(x)[h] = \sum_{i=1}^{n}\sum_{j=1}^{n} (h\, m_i)^T\, Z_{ij}(x)\, (h\, m_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} ({m_i}^T\, h)\, Z_{ij}\, (h\, m_j)$$

where m_i are monomials in x and $Z_{ij}(x)$ are polynomials in x.

Define $Z(x)$ as the N-by-N matrix of polynomials in x whose i, j^{th} element is Z_{ij}, and define $V(x)[h]$ as

$$V(x)[h]^T = h(m_1,\, m_2,\, \ldots,\, m_N).$$

We call Z the **middle matrix** for q and V its **border vector**. In this notation our representation is

$$q(x, h) = V(x)[h]^T Z(x) V(x)[h]$$

We can and typically do take $Z(x)$ to be symmetric. If the monomials m_i in $V(x)[h]$ do not repeat, then $Z(x)$ is uniquely determined and is symmetric.

Example 3.1. A "middle matrix" representation $g = 2$

$$3\, x_1 h x_2^2 h x_1 + h x_1 x_2 x_1 h - h x_1 h x_2^2 - x_2^2 h x_1 h + 5\, x_1 x_2 h x_2 h x_2 x_1$$
$$= \begin{pmatrix} h \\ hx_1 \\ hx_2x_1 \\ hx_2^2 \end{pmatrix}^T \begin{pmatrix} x_1x_2x_1 & 0 & 0 & -x_1 \\ 0 & 3\, x_2^2 & 0 & 0 \\ 0 & 0 & 5\, x_2 & 0 \\ -x_1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ hx_1 \\ hx_2x_1 \\ hx_2^2 \end{pmatrix}.$$

3.1.1 Positivity of q vs. Positivity of its Middle Matrix

A key fact is that positivity of q is equivalent to positivity of its middle matrix in the following sense.

Lemma 3.1. *Suppose $q(x)[h]$ a symmetric noncommutative polynomial in noncommuting variables pure quadratic in h and $Z(x)$ is its middle matrix. If $X \in (\mathbb{R}^{n\times n}_{sym})^g$ and $Z(X) \succeq 0$, then $q(X)[H] \succeq 0$ for all $H \in (\mathbb{R}^{n\times n}_{sym})^g$.*

Conversely, if $q(x)[h]$ is matrix positive; i.e., $q(X)[H] \succeq 0$, for every n and $X, H \in (\mathbb{R}^{n\times n}_{sym})^g$ in the (non empty) positivity domain $\{X :\ f(X) \succ 0\}$ of some polynomial f, then for each n and $X \in (\mathbb{R}^{n\times n}_{sym})^g$, we have $Z(X) \succ 0$ on $\{X :\ f(X) \succeq 0\}$.

Proof. The first statement is evident. The converse is proved in [CHSY03] in Lemma 9.5 and Theorem 10.10 in [CHSY03]. for a cleaner proofs see [HMV06] in particular Proposition 6.1. $\square$

3.2 The Zeroes Lemma

The following is useful in our analysis of subharmonics.

Lemma 3.2. *Let S be any $N \times N$ symmetric matrix with entries in $\mathbb{R}\langle x\rangle$. If there exists some diagonal entry $S_{ii} = 0$ and corresponding off-diagonal entries $S_{ij} = S_{ji}^T \neq 0$, then S is not matrix-positive semidefinite.*

Proof. Let e_i and e_j be standard basis vectors for R^N (i.e. $e_i^T A\, e_j = a_{ij}$) and define $v := \beta_1 e_i + \beta_2 e_j$ where $\beta_1,\ \beta_2 \in \mathbb{R}$. Then,

$$v^T S(x) v = ((S_{ij}\ +\ S_{ji})\,\beta_1 + S_{jj}\,\beta_2)\,\beta_2 = (2\,S_{ij}\,\beta_1 + S_{jj}\,\beta_2)\,\beta_2$$

Given $\beta_2 > 0$, we can choose β_1 such that

$$2\,\beta_1\,S_{ij}(X)\beta_2 + \beta_2 S_{jj}(X)\,\beta_2$$

is neither a positive nor negative matrix. □

This lemma is useful when applied to our matrix representation of the Laplacian of a symmetric NC polynomial .

3.3 The Laplacian of a Degree 4 Polynomial

We begin with a parameterization the set of degree 4 homogeneous polynomials in symmetric free variables

$$\begin{aligned} p = {} & A_1\,x_1^4\,x_1^2 + A_2(x_1^3\,x_2 + x_2\,x_1^3) + A_3\,(x_1^2\,x_2\,x_1 + x_1\,x_2\,x_1^2) + A_4\,(x_1^2\,x_2^2 + x_2^2\,x_1^2) \\ & + A_5\,(x_1\,x_2\,x_1\,x_2 + x_2\,x_1\,x_2\,x_1) + A_6\,x_1\,x_2^2\,x_1 + A_7\,(x_1\,x_2^3 + x_2^3\,x_1) + A_8\,x_2\,x_1^2\,x_2 \\ & + A_9\,(x_2\,x_1\,x_2^2 + x_2^2\,x_1\,x_2) + A_{10}\,x_2^4. \end{aligned}$$

We calculate the Laplacian of p:

$$\begin{aligned} & 2\,A_1\,(h^2\,x_1^2 + h\,x_1\,h\,x_1 + h\,x_1^2\,h + x_1\,h^2\,x_1 + x_1\,h\,x_1\,h + x_1^2\,h^2) \\ & + 2\,A_2\,(h^2\,x_1\,x_2 + h\,x_1\,h\,x_2 + x_1\,h^2\,x_2 + x_2\,h^2\,x_1 + x_2\,h\,x_1\,h + x_2\,x_1\,h^2) \\ & + 2\,A_3\,(h^2\,x_2\,x_1 + h\,x_1\,x_2\,h + h\,x_2\,h\,x_1 + h\,x_2\,x_1\,h + x_1\,h\,x_2\,h + x_1\,x_2\,h^2) \\ & + 2\,A_4\,(h^2\,x_1^2 + h^2\,x_2^2 + x_1^2\,h^2 + x_2^2\,h^2) \\ & + 2\,A_5\,(h\,x_1\,h\,x_1 + h\,x_2\,h\,x_2 + x_1\,h\,x_1\,h + x_2\,h\,x_2\,h) + \ 2\,A_6\,(h\,x_2^2\,h + x_1\,h^2\,x_1) \\ & + 2\,A_7\,(h^2\,x_2\,x_1 + h\,x_2\,h\,x_1 + x_1\,h^2\,x_2 + x_1\,h\,x_2\,h + x_1\,x_2\,h^2 + x_2\,h^2\,x_1) + \ 2\,A_8\,(h\,x_1^2\,h + x_2\,h^2\,x_2) \\ & + 2\,A_9\,(h^2\,x_1\,x_2 + h\,x_1\,h\,x_2 + h\,x_1\,x_2\,h + h\,x_2\,x_1\,h + x_2\,h\,x_1\,h + x_2\,x_1\,h^2) \\ & + 2\,A_{10}\,(h^2\,x_2^2 + h\,x_2\,h\,x_2 + h\,x_2^2\,h + x_2\,h^2\,x_2 + x_2\,h\,x_2\,h + x_2^2\,h^2). \end{aligned}$$

The directional Laplacian is quadratic in h, and so can be represented by border vector

$$V(x)[h]^T = \begin{pmatrix} h & x_1\,h & x_2\,h & x_1^2\,h & x_1\,x_2\,h & x_2\,x_1\,h & x_2^2 h \end{pmatrix}^T$$

and middle matrix $Z(x)$ which is

$$\begin{pmatrix} \begin{matrix}(A_1+A_8)\,x_1^2+(A_6+A_{10})\,x_2^2\\ +(A_3+A_9)\,(x_1\,x_2+x_2\,x_1)\end{matrix} & \begin{matrix}(A_1+A_5)\,x_1\\ +(A_3+A_7)\,x_2\end{matrix} & \begin{matrix}(A_2+A_9)\,x_1\\ +(A_5+A_{10})\,x_2\end{matrix} & A_1{+}A_4 & A_3{+}A_7 & A_2{+}A_9 & A_4{+}A_{10} \\ \begin{matrix}(A_1+A_5)\,x_1\\ +(A_3+A_7)\,x_2\end{matrix} & A_1+A_6 & A_2+A_7 & 0 & 0 & 0 & 0 \\ \begin{matrix}(A_2+A_9)\,x_1\\ +(A_5+A_{10})\,x_2\end{matrix} & A_2+A_7 & A_8+A_{10} & 0 & 0 & 0 & 0 \\ A_1+A_4 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 \\ A_3+A_7 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ A_2+A_9 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 \\ A_4+A_{10} & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \end{pmatrix}.$$

Assume that $Z(X)$ is a positive semidefinite matrix for $X \in (\mathbb{R}^{n\times n}_{sym})^g$. By the Zeroes Lemma, the zeroes on the last four diagonals force all entries in the last four rows or columns to be zero, that is:

$$A_4 = -A_1, \quad A_{10} = A_1, \quad A_9 = -A_2, \quad A_7 = -A_3. \tag{15}$$

Applying these conditions to the matrix above, and ignoring the rows and columns which are zero, we have:

$$\begin{pmatrix} \begin{matrix}(A_1+A_8)\,x_1^2+(A_1+A_6)\,x_2^2\\ +(A_2-A_3)(x_2\;x_1-x_1\;x_2)\end{matrix} & (A_1{+}A_5)x_1 & (A_1{+}A_5)x_2 \\ (A_1{+}A_5)x_1 & A_1{+}A_6 & A_2{-}A_3 \\ (A_1{+}A_5)x_2 & A_2{-}A_3 & A_1{+}A_8 \end{pmatrix}.$$

This matrix can be simplified by substitution of reoccurring pairs by single letters:

$$G = A_1 + A_6, \quad H = A_1 + A_8 \quad J = A_2 - A_3, \quad K = A_1 + A_5.$$

to obtain

$$\begin{pmatrix} Hx_1^2 - J(x_1\,x_2 + x_2\,x_1) + Gx_2^2 & Kx_1 & Kx_2 \\ Kx_1 & G & J \\ Kx_2 & J & H \end{pmatrix}.$$

We now find its noncommutative LDL^T (Cholesky) decomposition to have D term equal to

$$\begin{pmatrix} G & 0 & 0 \\ 0 & H - \frac{J^2}{G} & 0 \\ 0 & 0 & Hx_1^2 + J(x_1\,x_2 + x_2\,x_1) + Gx_2^2 - \frac{K^2x_1^2}{G} - \frac{(\frac{JKx_1}{G}+Kx_2)\,(\frac{JKx_1}{G}+Kx_2)}{H-\frac{J^2}{G}} \end{pmatrix}.$$

A reference is [CHSY03] which describes the NCAlgebra command, NCLDUDecomposition, we used to do this.

We see there are three inequalities, which must be satisfied for $Z(X)$ to be positive semidefinite.

$$G > 0, \quad H - \frac{J^2}{G} > 0,$$

$$HX_1^2 + J(X_1\,X_2 + X_2\,X_1) + GX_2^2 - \frac{K^2X_1^2}{G} - \frac{(\frac{JKX_1}{G} + KX_2)\,(\frac{JKX_1}{G} + KX_2)}{H - \frac{J^2}{G}} > 0.$$

The last condition is purely quadratic in X_1 and X_2, and therefore has a middle matrix representation which we compute to be

$$\begin{pmatrix} G - \frac{H^2K^2}{(G-\frac{H^2}{J})J^2} - \frac{K^2}{J} & H + \frac{HK^2}{(G-\frac{H^2}{J})J} \\ H + \frac{HK^2}{(G-\frac{H^2}{J})J} & J - \frac{K^2}{G-\frac{H^2}{J}} \end{pmatrix}.$$

Again we perform the LDL^T decomposition:

$$\begin{pmatrix} G - \frac{K^2}{H-\frac{J^2}{JG}} & 0 \\ 0 & H - \frac{J^2K^2}{(H-\frac{J^2}{G})G^2} - \frac{K^2}{G} - \frac{\left(J-\frac{JK^2}{(H-\frac{J^2}{G})G}\right)^2}{G-\frac{K^2}{H-\frac{J^2}{G}}} \end{pmatrix}.$$

Although the inequality

$$H - \frac{J^2K^2}{(H - \frac{J^2}{G})G^2} - \frac{K^2}{G} - \frac{\left(J - \frac{JK^2}{(H-\frac{J^2}{G})G}\right)^2}{G - \frac{K^2}{H-\frac{J^2}{G}}} > 0 \tag{16}$$

is quite complicated, we can simplify it some by multiplying it by expressions which are known to be positive, such as:

$$G, \quad H - \frac{J^2}{G}, \quad \text{and} \quad G - \frac{K^2}{H - \frac{J^2}{G}}$$

which we encountered earlier. This gives a polynomial inequality equivalent to (16), which, after some simplification, gives us:

$$(GH - J^2 - K^2)^2 > 0.$$

Which, considering only the case of all real coefficients, is rather vacuous, informing us only that $GH - J^2 - K^2 \neq 0$.

Bringing all our inequalities together (simplifying each as we did above), we obtain

$$(I)\;G > 0, \qquad (II)\;GH > J^2, \qquad (III)\;GH > J^2 + K^2, \qquad (IV)\;GH \neq H^2 + K^2.$$

Notice (III) implies (II) and (IV), thus reducing to (I) and (II). Therefore, we conclude that the set of polynomials making the Laplacian matrix "positive" is exactly those of the form:

$$\begin{aligned} f \;\; = \;\; & A_1(x_1^4 - x_1^2\,x_2^2 - x_2^2\,x_1^2 + x_2^4) + A_2(x_1^3\,x_2 + x_2\,x_1^3 - x_2\,x_1\,x_2^2 - x_2^2\,x_1\,x_2) \\ & + A_3(x_1^2\,x_2\,x_1 + x_1\,x_2\,x_1^2 - x_1\,x_2^3 - x_2^3\,x_1) + A_5(x_1\,x_2\,x_1\,x_2 + x_2\,x_1\,x_2\,x_1) \\ & + A_6\,x_1\,x_2^2\,x_1 + A_8 x_2\,x_1^2\,x_2 \end{aligned} \tag{17}$$

with coefficients satisfying the inequalities:

$$(III) \implies (A_1 + A_8)(A_1 + A_6) > (A_3 - A_2)^2 + (A_1 + A_5)^2 \text{ and} \tag{18}$$

$$(I) \implies \quad A_1 + A_8 > 0 \quad (\text{or, equivalently} \quad A_1 + A_6 > 0). \tag{19}$$

For neatness, and to more clearly see the dimension of the space of subharmonics,we set $B_1 = A_1, B_2 = A_2, B_3 = A_3, B_4 = A_5, B_5 = A_6, B_6 = A_8$. □

4 Uniqueness Proofs

Now we set about to prove that the list of subharmonic and harmonic polynomials in Theorem 1 is complete. We do this, as is required, only for two variables but in the course of our proof we discover some promising recursions valid in any number of variables.

4.1 Even Degree Homogeneous p

Given $0 < m < d$ a noncommutative polynomial p of degree d decompose it as

$$p = \sum_{|t|=m} x^t p_t(x) + \Lambda \tag{20}$$

where $deg\ \Lambda < m$. Call the polynomial $p_t(x)$ the **right neighbor of** x^t

Lemma 4.1. *If p is harmonic in any number of variables consider the right neighbor representation of p for any m; the right neighbor p_t of each monomial x^t of degree m is harmonic, that is, $Lap(p_t) = 0$.*

If p is subharmonic in any number of variables, if p is homogeneous of degree d then the right neighbor p_t of each monomial x^t of degree $\frac{d}{2}$ is harmonic, that is, $Lap(p_t) = 0$.

Proof. Apply the Laplacian to the right neighbor decomposition (20) of p and get from the product rule for the Laplacian (Lemma 2.2):

$Lap[p, h] =$

$$\sum_{|t|=m} x^t Lap[p_t(x), h] \tag{21}$$

$$+ \sum_{|t|=m} Lap[x^t, h] p_t(x) \tag{22}$$

$$+ 2 \sum_{|t|=m} D(x^t)[h] D(p_t)[h] \tag{23}$$

$$+ Lap[\Lambda, h].$$

Suppose $Lap[p, h] = 0$. We shall now show that polynomial (21) is 0, (22) is 0, and (23) is 0. All terms of the polynomials (21), (22), and (23) have degree at least m, while $deg\ \Lambda < m$. Since the Laplacian of a polynomial respects degree, we have $Lap(\Lambda) = 0$. Next factor a given degree $\geq m$ monomial r into its *$m-$front and back:* namely, $r = r_f r_b$ where r_f has degree m. Consider the polynomial (21): the $m-$back of each monomial in it contains two h's. Likewise the $m-$back of each monomial in (22) and (23) contains no h's and one h respectively. Thus polynomials (21), (22), and (23) contain no monomials which cancel and since their sum is zero they must be zero. From (21) is 0 we immediately get $Lap(p_t) = 0$. This proves the first part of the lemma.

Now to the subharmonic part. That $Lap[p, h]$ is matrix positive implies that it is a sum of squares:

$$Lap(p) = \sum_j L_j^T(x)[h] L_j(x)[h] \tag{24}$$

First observe that each L_j is linear in h. This is true since the highest degree in h monomial $\lambda(x)[h]$ of $L_j(x)[h]$ contributes a $\lambda(x)[h]^T \lambda(x)[h]$ to $L_j^T(x)[h] L_j(x)[h]$ monomial which holds because its coefficient is positive and can not be cancelled out; likewise $\lambda(x)[h]^T \lambda(x)[h]$ appears in $Lap(p)$. Thus the monomial $\lambda(x)[h]$ has degree one in h.

Because of equation (24) we can refer to each term of $Lap(p)$ as having a first half and second half; each half has degree $\frac{d}{2}$. Also every term of $Lap(p)$ has an h in its first half and also in its second half. However, if $m = \frac{d}{2}$ all terms in (21) have two h's in their second half and none in their first half. This contradicts the previous sentence; thus equation (21) is 0. Since we have factored out x^t in the representation (21), their coefficients $Lap(p_t)$ are 0 for all $|t| = \frac{d}{2}$. $\square$

4.1.1 Homogeneous Subharmonics are Sums of Products of Harmonics

In this section we prove under weak hypotheses that homogeneous subharmonics are sums of products of harmonics. A subharmonic polynomial of odd degree is harmonic, so is the product of itself and 1. Thus we restrict to even degree and prove the following.

Proposition 4.1. *Assume the harmonic polynomials homogeneous of degree $\frac{d}{2}$ are the span of $\gamma_1, \cdots, \gamma_k$. Assume there is a monomial w_j in γ_j which does not occur in the other $\gamma_1, \cdots, \gamma_k$. If p is subharmonic homogeneous of even degree d, then it has the form*

$$p = \sum_{i,j=1}^{k} \phi_{ij}\gamma_i\gamma_j \tag{25}$$

where each ϕ_{ij} is a real number. Note further that for symmetric p we may take $\phi_{ij} = \phi_{ji}$. Let $\mathcal{S}$ denote the span of these symmetric subharmonics.

This implies that $\mathcal{S}$ is a space of at most dimension $\frac{k(k+1)}{2}$. For example, in two variables there are two independent homogeneous harmonic polynomials of degree other than 2, so dim $\mathcal{S}$ is at most 3 for all $d \neq 4$. For $d = 4$ we have dim $\mathcal{S} \leq 6$.

Proof. Assume p is subharmonic homogeneous of degree d. Write down its right neighbor representation with $m = \frac{d}{2}$ and use Lemma 4.1 to get $Lap(p_t) = 0$ for $|t| = \frac{d}{2}$. Thus

$$p_t = \sum_{j}^{k} \mu_j(t)\gamma_j$$

for some numbers $\mu_j(t)$. Plug this into the decomposition

$$p = \sum_{|t|=\frac{d}{2}} x^t p_t(x) \ = \sum_{|t|=\frac{d}{2}} \sum_j \mu_j(t) x^t \gamma_j \tag{26}$$

to get

$$p = \sum_j \left(\sum_{|t|=\frac{d}{2}} \mu_j(t) x^t \right) \gamma_j = \sum_j^k p^j(x)\gamma_j.$$

Now make a left neighbor decomposition of p which by the definition of the monomial w_1 has the form

$$p^1(x)w_1 + G$$

where all terms of G are without w_1 on the right. The left neighbor version of Lemma 4.1 implies $Lap(p^1(x)) = 0$. Likewise each $p^j(x)$ is harmonic of degree $\frac{d}{2}$. This proves representation (25) for p. $\square$

Next we prove our representation of subharmonics stated in the introduction as Theorem 2.

Proof of Theorem 2

Now suppose p is symmetric. Proposition 4.1 says we can represent p as in equation (25). Note if u is harmonic then u^T is harmonic and relabel and possibly expand (by taking transposes) the set $\gamma_1, \dots, \gamma_k$ as

$$s_1, \dots, s_\alpha, u_1, \dots, u_\beta, u_1^T, \dots, u_\beta^T$$

where the s_i are symmetric polynomials. Set $\Psi := \{\tilde{\phi}_{ij}\}_{i,j=1}^{\alpha+2\beta}$ where $\tilde{\phi}_{ij} = \phi_{ij}$ for i, j corresponding to an original γ_ℓ and 0 otherwise. Now let $s = \begin{pmatrix} s_1 \\ \vdots \\ s_\alpha \end{pmatrix}$, $u = \begin{pmatrix} u_1 \\ \vdots \\ u_\beta \end{pmatrix}$ and $v = \begin{pmatrix} u_1^T \\ \vdots \\ u_\beta^T \end{pmatrix}$. Then $p = \begin{pmatrix} s \\ u \\ v \end{pmatrix}^T \Psi \begin{pmatrix} s \\ u \\ v \end{pmatrix}$ and

$$p = \frac{p + p^T}{2} = \begin{pmatrix} s \\ u \\ v \end{pmatrix}^T \Phi \begin{pmatrix} s \\ u \\ v \end{pmatrix}$$

where $\Phi = \dfrac{\Psi + \Psi^T}{2}$, a symmetric matrix as required.

Decompose the symmetric matrix Φ as $\Phi = NJN^T$ where J is a diagonal matrix with ± 1 or 0 on the diagonal and N has real numbers as entries. Now, let us put $R = N^T \begin{pmatrix} s \\ u \\ v \end{pmatrix}$. Then

$$p = \begin{pmatrix} s \\ u \\ v \end{pmatrix}^T \Phi \begin{pmatrix} s \\ u \\ v \end{pmatrix} = \begin{pmatrix} s \\ u \\ v \end{pmatrix}^T NJN^T \begin{pmatrix} s \\ u \\ v \end{pmatrix} = R^T JR = \sum_i c_i R_i^T R_i.$$

where c_j is ± 1 or 0. The $s_i,\ u_i,\ u_i^T$ are harmonic, so their linear combinations R_i are harmonic. $\square$

An appealing, easily proved, formula is

$$Lap(p) = 2 \sum_j^g D[R, x_j, h]^T J D[R, x_j, h].$$

Clearly, if the matrix $\Phi := \{\phi_{ij}\}_{i,j=1}^{\alpha+2\beta}$ is positive semidefinite (or equivalently J has nonnegative entries), $Lap(p)$ will be positive, so then p is subharmonic. Also we get even degree harmonics are sums and differences of squares of harmonics. It is not clear which differences make p harmonic or subharmonic.

However, we conjecture

A homogeneous symmetric subharmonic polynomial p of even degree d is a finite sum

$$p = \sum_i^{finite} R_i^T R_i + \sum_\ell^{finite} H_\ell$$

for some homogeneous harmonic functions R_i, H_ℓ with R_i of degree $\frac{d}{2}$ and H_ℓ of degree d.

At this point, we have finished discussing subharmonics, and will now turn our full attention to harmonic polynomials.

4.2 Uniqueness of Harmonics in Two Variables

4.2.1 Polynomials of degree Three and Larger

At this point, we have proved that there are harmonic polynomials of arbitrary degree. Working in two variables, we will now show that the polynomials $\operatorname{Re}\gamma^d$ and $\operatorname{Im}\gamma^d$ span all of the harmonics. This can be a helpful result, which, as yet, we have been unable to show for any higher number of variables. In fact, McAllaster has found, experimentally, that in three variables, the size of the basis of harmonic polynomials increases on the order of d^2 (See [McA04]).

Proposition 4.2. *Let $\gamma = x_1 + ix_2$, and $\mathcal{B}_d = \{\operatorname{Re}\gamma^d, \operatorname{Im}\gamma^d\}$. Then $\mathcal{B}_d$ forms a basis for all harmonic polynomials which are homogeneous of degree d for any $d > 2$.*

To prove the proposition we need two lemmas.

Lemma 4.2. *In degree three, there are two linearly independent homogeneous harmonic polynomials whose span is all harmonic polynomials which are homogeneous of degree three.*

Lemma 4.3. *Let $\beta(x_1, x_2)$ be harmonic and homogeneous of degree d. Then we may uniquely represent β as $\beta(x_1, x_2) = x_1 f(x_1, x_2) + x_2 g(x_1, x_2)$, where f and g are harmonic and homogeneous of degree $d-1$ and $D[f(x_1, x_2), x_1, h] = -D[g(x_1, x_2), x_2, h]$.*

Proof. **(Lemma 4.2)** Every homogeneous polynomial of degree three has the form

$$a_1x_1^3 + a_2x_1^2x_2 + a_3x_1x_2x_1 + a_4x_1x_2^2 + a_5x_2x_1^2 + a_6x_2x_1x_2 + a_7x_2^2x_1 + a_8x_2^3$$

and the Laplacian of this is

$$a_1h^2x_1 + a_7h^2x_1 + a_2h^2x_2 + a_8h^2x_2 + a_1x_1h^2 + a_4x_1h^2$$

$$+ a_5x_2h^2 + a_8x_2h^2 + a_1hx_1h + a_6hx_1h + a_3hx_2h + a_8hx_2h$$

$$= (a_1 + a_7)h^2x_1 + (a_2 + a_8)h^2x_2 + (a_1 + a_4)x_1h^2 + (a_5 + a_8)x_2h^2 + (a_1 + a_6)hx_1h + (a_3 + a_8)hx_2h,$$

so if we want the polynomial to be harmonic, we need each monomial of the Laplacian to be zero, so we need the equations $a_1 + a_7 = 0,\ a_2 + a_8 = 0,\ a_1 + a_4 = 0,\ a_5 + a_8 = 0,\ a_1 + a_6 = 0,\ a_3 + a_8 = 0$ to hold. This amounts to having the vector $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ in the nullspace of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which has the basis $\{(0, -1, -1, 0, -1, 0, 0, 1), (-1, 0, 0, 1, 0, 1, 1, 0)\}$, corresponding to polynomials $x_2^3 - x_1^2x_2 - x_2x_1^2 - x_1x_2x_1$ and $-x_1^3 + x_1x_2^2 + x_2^2x_1 + x_2x_1x_2$. Hence there are exactly two linearly independent harmonic polynomials which are homogeneous of degree three. □

Proof. **(Lemma 4.3)** Now, suppose that we are given a polynomial function $\beta(x_1, x_2)$ which is harmonic, homogeneous, and of degree d. Then, every monomial of β begins with either x_1 or x_2, so we may uniquely represent β by the neighbor decomposition $\beta(x_1, x_2) = x_1 f(x_1, x_2) + x_2 g(x_1, x_2)$.

Now, by Lemma 4.1, we know that f and g are harmonic, and using the product rule for the Laplacian, we find:

$$\begin{aligned} Lap[\beta, h] &= Lap[x_1 f(x_1,x_2) + x_2 g(x_1,x_2), h] \\ &= (Lap[x_1,h]f(x_1,x_2) + x_1 Lap[f(x_1,x_2),h] \ + \ 2(D[x_1,x_1,h]D[f(x_1,x_2),x_1,h] \\ &\quad +D[x_1,x_2,h]D[f(x_1,x_2),x_2,h])) + (Lap[x_2,h]g(x_1,x_2) + x_2 Lap[g(x_1,x_2),h] \\ &\quad +2(D[x_2,x_1,h]D[g(x_1,x_2),x_1,h] + D[x_2,x_2,h]D[g(x_1,x_2),x_2,h])) \\ &= 2h(D[f(x_1,x_2),x_1,h] + D[g(x_1,x_2),x_2,h]) \end{aligned}$$

and since β is harmonic, this is zero, which gives:

$$0 = 2h(D[f(x_1,x_2),x_1,h] + D[g(x_1,x_2),x_2,h]).$$

or more specifically,

$$D[f(x_1,x_2),x_1,h] = -D[g(x_1,x_2),x_2,h]. \tag{27}$$

□

Proof. **(Prop. 4.2)** First of all, we show that $\operatorname{Re}\gamma^d$ and $\operatorname{Im}\gamma^d$ are linearly independent:

Suppose they are linearly dependent. Then $a\operatorname{Re}\gamma^d = b\operatorname{Im}\gamma^d$, where $a \neq 0$, $b \neq 0$, $a, b \in \mathbb{R}$. But then, $\gamma^d = (x_1+ix_2)(R+iI)$, where $R = \operatorname{Re}\gamma^{d-1}$, $I = \operatorname{Im}\gamma^{d-1}$, so that $a(x_1R-x_2I) = b(x_1I+x_2R)$. Now, we equate the terms starting with x_1 and x_2, respectively, to get that $aR = bI$ and $-aI = bR$. Then, we get that $R = (b/a)I$ and $I = (-b/a)R$ from the first and second equations, repectively. Puting this together, we get $R = (b/a)I = (b/a)(-b/a)R$, so that cancelling R, we get $b^2/a^2 = -1$, or $b^2 = -a^2$, which can happen only if $a = b = 0$, a contradiction.

Now, we are going to prove the proposition by induction.

First of all, Lemma 4.2 begins the induction. To prove the rest of the proposition, suppose that for degree $d-1 \geq 3$, we know that there are exactly two linearly independent polynomials which are harmonic and homogeneous. Then for degree d, we suppose that β is harmonic and homogeneous. Then

$$\beta(x_1,x_2) = x_1\varphi(x_1,x_2) + x_2\psi(x_1,x_2),$$

where φ and ψ are both harmonic, and homogeneous of degree $d-1$ (Lemma 4.3). Then by the induction hypothesis

$$\beta(x_1,x_2) = x_1(a_\varphi \operatorname{Re}\gamma^{d-1} + b_\varphi \operatorname{Im}\gamma^{d-1}) + x_2(a_\psi \operatorname{Re}\gamma^{d-1} + b_\psi \operatorname{Im}\gamma^{d-1}),$$

but from Lemma 4.3, we know that we must have $D[\varphi,x_1,h] + D[\psi,x_2,h] = 0$ which is equivalent to saying that

$$a_\varphi D[\operatorname{Re}\gamma^{d-1},x_1,h] + b_\varphi D[\operatorname{Im}\gamma^{d-1},x_1,h] + a_\psi D[\operatorname{Re}\gamma^{d-1},x_2,h] + b_\psi D[\operatorname{Im}\gamma^{d-1},x_2,h] = 0.$$

Now, by applying the identities for the derivatives of $\operatorname{Re}\gamma^d$ and $\operatorname{Im}\gamma^d$ (see Lemma 2.3), we get the following:

$$a_\varphi D[\operatorname{Re}\gamma^{d-1},x_1,h] + b_\varphi D[\operatorname{Im}\gamma^{d-1},x_1,h] - a_\psi D[\operatorname{Im}\gamma^{d-1},x_1,h] + b_\psi D[\operatorname{Re}\gamma^{d-1},x_1,h] = 0,$$

so

$$(a_\varphi + b_\psi)D[\operatorname{Re}\gamma^{d-1},x_1,h] + (b_\varphi - a_\psi)D[\operatorname{Im}\gamma^{d-1},x_1,h] = 0,$$

which gives that

$$0 = \quad D[(a_\varphi + b_\psi)\operatorname{Re}\gamma^{d-1} + (b_\varphi - a_\psi)\operatorname{Im}\gamma^{d-1}, x_1, h],$$

but if the derivative of a function with respect to x_1 is zero, that function must be a polynomial in x_2, but we know that $\operatorname{Re}\gamma^{d-1}$ and $\operatorname{Im}\gamma^{d-1}$ are homogeneous of degree $d-1$, so the function of x_2 can only be cx_2^{d-1} for some constant c. That is to say

$$(a_\varphi + b_\psi)\operatorname{Re}\gamma^{d-1} + (b_\varphi - a_\psi)\operatorname{Im}\gamma^{d-1} = cx_2^{d-1}.$$

Now since x_2^{d-1} is not harmonic, it is not in the span of $\operatorname{Re}\gamma^{d-1}$ and $\operatorname{Im}\gamma^{d-1}$, so $c = 0$. Therefore,

$$0 = (a_\varphi + b_\psi)\operatorname{Re}\gamma^{d-1} + (b_\varphi - a_\psi)\operatorname{Im}\gamma^{d-1},$$

which means that $0 = a_\varphi + b_\psi$ and $0 = b_\varphi - a_\psi$, so

$$a_\varphi = -b_\psi \text{ and } a_\psi = b_\varphi.$$

Using this, we get

$$\begin{aligned}
\beta(x_1, x_2) &= x_1(a_\varphi \operatorname{Re}\gamma^{d-1} + b_\varphi \operatorname{Im}\gamma^{d-1}) + x_2(a_\psi \operatorname{Re}\gamma^{d-1} + b_\psi \operatorname{Im}\gamma^{d-1}) \\
&= x_1(a_\varphi \operatorname{Re}\gamma^{d-1} + a_\psi \operatorname{Im}\gamma^{d-1}) + x_2(a_\psi \operatorname{Re}\gamma^{d-1} - a_\varphi \operatorname{Im}\gamma^{d-1}) \\
&= a_\varphi x_1 \operatorname{Re}\gamma^{d-1} + a_\psi x_1 \operatorname{Im}\gamma^{d-1} + a_\psi x_2 \operatorname{Re}\gamma^{d-1} - a_\varphi x_2 \operatorname{Im}\gamma^{d-1} \\
&= a_\varphi(x_1 \operatorname{Re}\gamma^{d-1} - x_2 \operatorname{Im}\gamma^{d-1}) + a_\psi(x_1 \operatorname{Im}\gamma^{d-1} + x_2 \operatorname{Re}\gamma^{d-1}) \\
&= a_\varphi \operatorname{Re}\gamma^d + a_\psi \operatorname{Im}\gamma^d,
\end{aligned}$$

which implies that β is linearly dependent upon $\operatorname{Re}\gamma^d$ and $\operatorname{Im}\gamma^d$. Hence $\operatorname{Re}\gamma^d$ and $\operatorname{Im}\gamma^d$ form a basis for all of the harmonic polynomials which are homogeneous of degree d.

□

4.2.2 Degree Two Polynomials

The polynomials of degree two are a special case. This is because some terms of polynomials will vanish when the Laplacian is taken. Specifically, if we are given the general polynomial

$$p = A_1x_1^2 + A_2x_2^2 + A_3x_1x_2 + A_4x_2x_1,$$

we find that the Laplacian is

$$Lap(p) = A_1h^2 + A_2h^2,$$

meaning that the polynomial will be harmonic provided that $A_1 + A_2 = 0$ and subharmonic provided that $A_1 + A_2 \geq 0$. This is the one case where the harmonic polynomial is not symmetric. Also, because the subharmonic polynomials are built up of harmonic polynomials of one half the degree, this means that it may be possible, in the degree four case alone, to create nonsymmetric subharmonic polynomials.

Remark 2. We now show that this gives a 6 dimensional spanning set for the symmetric subharmonics of degree 4; denote these by $\mathcal{S}_4$. We use Proposition 4.1 which says $\mathcal{S}_4$ is spanned by symmetrized products of the basis

$$s =: x_1^2 - x_2^2, u := x_1x_2, u^T := x_2x_1$$

. Thus we obtain

$$s^2, su + u^Ts, su^T + us, uu + u^Tu^T, u^Tu, uu^T.$$

Note this is consistent with Theorem 1 part (2b) which implies the span of the degree 6 symmetric subharmonics has dimension.

4.3 Homogeneous Harmonics of Odd Degree

The remainder of this section is not used in the rest of the paper, but Proposition 4.3 may be useful in further research on harmonics in many variables.

What does the argument in Section 4.1.1 say about harmonics of odd degree? As we have already stated in §2.4, any subharmonic polynomial of odd degree is required to be harmonic.

Given NC polynomial p decompose it as

$$p = \sum_{|t|=\frac{d-1}{2}} \sum_{i=1}^{g} x^t x_i p_{t,i}(x) \tag{28}$$

and call the polynomial $p_{t,i}(x)$ the **right neighbor of** $x^t x_i$. Here we are assuming all terms of p have degree $\frac{d-1}{2}$.

Apply Lap to the right neighbor decomposition (28) of harmonic p and from the Laplacian Product Rule get
$0 = Lap[p, h] =$

$$\sum_{|t|=\frac{d-1}{2}} \sum_{i=1}^{g} Lap[x^t x_i, h] p_{t,i}(x) \tag{29}$$

$$+ \sum_{|t|=\frac{d-1}{2}} \sum_{i=1}^{g} x^t x_i Lap[p_{t,i}(x), h] \tag{30}$$

$$+ 2 \sum_{|t|=\frac{d-1}{2}} \sum_{i=1}^{g} \sum_{j=1}^{g} D[x^t x_i, x_j, h] D[p_{t,i}(x), x_j, h] \tag{31}$$

which is

$$\sum_{|t|=\frac{d-1}{2}} \sum_{i=1}^{g} \left(Lap[x^t, h] x_i + x^t Lap[x_i, h] + 2 \sum_{j=1}^{g} D[x^t, x_j, h] D[x_i, x_j, h] \right) p_{t,i}(x) \tag{32}$$

$$+ \sum_{|t|=\frac{d-1}{2}} \sum_{i=1}^{g} x^t x_i Lap[p_{t,i}(x), h] \tag{33}$$

$$+ 2 \sum_{|t|=\frac{d-1}{2}} \sum_{i=1}^{g} \sum_{j=1}^{g} \left(D[x^t, x_j, h] x_i + x^t D[x_i, x_j, h] \right) D[p_{t,i}(x), x_j, h] \tag{34}$$

Finally it becomes

$$\sum_{|t|=\frac{d-1}{2}} \sum_{i=1}^{g} \left(Lap[x^t, h] x_i + 2D[x^t, x_i, h] h \right) p_{t,i}(x) \tag{35}$$

$$+ \sum_{|t|=\frac{d-1}{2}} \sum_{i=1}^{g} x^t x_i Lap[p_{t,i}(x), h] \tag{36}$$

$$+ 2 \sum_{|t|=\frac{d-1}{2}} \sum_{i}^{g} \left(x^t h D[p_{t,i}(x), x_i, h] + \sum_{j=1}^{g} D[x^t, x_j, h] x_i D[p_{t,i}(x), x_j, h] \right), \tag{37}$$

which must be 0. The right half of each monomial in (35) contains no h's, while in (36) and (37) each right half does; thus no term of (35) can be cancelled. We conclude (35) is 0. Similarly the right halves in (36) are the only right halves monomials which contain two h's and so cannot be cancelled. Thus (36) is 0 and so we get

$$Lap[p_{t,i}, h] = 0 \text{ for each } |t| = \tfrac{d-1}{2}, i = 1, \cdots, g.$$

Use

$$p_{t,i} = \sum_{j}^{k} \mu_j(t,i)\gamma_j$$

as before. Plug this into the decomposition

$$p = \sum_{|t|=\frac{d-1}{2},i} x^t x_i p_{t,i}(x) = \sum_{|t|=\frac{d-1}{2},i} \sum_j \mu_j(t,i) x^t x_i \gamma_j. \tag{38}$$

to get

$$p = \sum_j \left(\sum_{|t|=\frac{d-1}{2},i} \mu_j(t,i) x^t x_i \right) \gamma_j = \sum_j \left(\sum_i \left(\sum_{|t|=\frac{d-1}{2}} \mu_j(t,i) x^t \right) x_i \right) \gamma_j$$

$$= \sum_{j=1}^{k} \left(\sum_i p^{j,i}(x) x_i \right) \gamma_j$$

Now make a left neighbor decomposition of p which by the definition γ_1 has the form

$$\left(\sum_i p^{1,i}(x) x_i \right) \gamma_1 + G$$

where all terms of G are without γ_1 on the right. The left handed version of Lemma 4.1 implies $Lap(p^{1,i}) = 0$ for each $i = 1, \cdots, g$.

We have proved the following:

Proposition 4.3. *Assume the harmonic polynomials homogeneous of degree $\frac{d-1}{2}$ are the span of $\gamma_1, \cdots, \gamma_k$.*
Assume there is a monomial w_j in γ_j which does not occur in the other $\gamma_1, \cdots, \gamma_k$.
If p is subharmonic homogeneous of odd degree d, then it is harmonic and has the form

$$p = \sum_{i=1}^{g} \sum_{m,j=1}^{k} \phi_{mij} \gamma_m x_i \gamma_j \tag{39}$$

where each ϕ_{mij} is a number.

Question 1. What ϕ_{mij} make it 0? That is to say, what properties must ϕ_{mij} satisfy in order for p to be harmonic?

We do a few calculations which might someday help with this question. Note the Laplacian of such a p is:

$$Lap[p,h] = \sum_{i=1}^{g} \sum_{m,j=1}^{k} \phi_{mij} Lap[\gamma_m x_i \gamma_j, h]$$

$$= \sum_{i=1}^{g} \sum_{m,j=1}^{k} \phi_{mij} \Big(Lap[\gamma_m, h] x_i \gamma_j + \gamma_m Lap[x_i, h] \gamma_j + \gamma_m x_i Lap[\gamma_j, h]$$

$$+ 2 \sum_{l=1}^{g} (\gamma_m D[x_i, x_l, h] D[\gamma_j, x_l, h] + D[\gamma_m, x_l, h] x_i D[\gamma_j, x_l, h] + D[\gamma_m, x_l, h] D[x_i, x_l, h] \gamma_j) \Big)$$

$$= 2 \sum_{i=1}^{g} \sum_{m,j=1}^{k} \phi_{mij} \Big(\gamma_m h D[\gamma_j, x_i, h] + D[\gamma_m, x_i, h] h \gamma_j + \sum_{l=1}^{g} D[\gamma_m, x_l, h] x_i D[\gamma_j, x_l, h] \Big)$$

As before cancellation cannot occur between terms with right halves containing two h's, one h and no h's. Thus, Lap$[p, h] = 0$ is equivalent to

$$\sum_{m=1}^{k} \gamma_m h \sum_{i=1}^{g} D\left[\sum_{j=1}^{k} \phi_{mij} \gamma_j, x_i, h\right] = 0, \quad \text{and} \quad \sum_{j=1}^{k} \left[\sum_{i=1}^{g} D\left[\sum_{m=1}^{k} \phi_{mij} \gamma_m, x_i, h\right]\right] h \gamma_j = 0 \qquad \text{and}$$

$$\sum_{\ell=1}^{g} \sum_{i=1}^{g} \sum_{j=1}^{k} D\left[\sum_{m}^{k} \phi_{mij} \gamma_m, x_l, h\right] x_i D\left[\gamma_j, x_l, h\right] = 0.$$

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